Problem 1.11

Solve $y'' + (x+2)y' + (1+x)y = 0$.

Solution

This ODE will be solved with reduction of order, also known as multiplicative substitution. This involves finding a solution by inspection, making a prescribed substitution, and solving the reduced, simplified ODE. Because the equation is second order, we expect there to be two linearly independent solutions and two arbitrary constants in the solution. Notice that the coefficients of the odd derivatives of y add to the same value that the coefficients of the even derivatives of y do, namely $x+2$. Thus, one solution to this ODE is $y_1(x) = e^{-x}$. Let's verify this.

$$\begin{aligned} y_1'' + (x+2)y_1' + (1+x)y_1 &= e^{-x} - (x+2)e^{-x} + (1+x)e^{-x} \\ &= -(x+2)e^{-x} + (2+x)e^{-x} \\ &= 0 \end{aligned}$$

The next step is to make the prescribed substitution, $y = y_1 u(x)$, into the ODE. Find expressions for y' and y'' in terms of the new variable u.

$$\begin{aligned} y &= e^{-x}u(x) \\ y' &= -e^{-x}u + e^{-x}u' = e^{-x}(u' - u) \\ y'' &= -e^{-x}(u' - u) + e^{-x}(u'' - u') = e^{-x}(u'' - 2u' + u) \end{aligned}$$

Now plug these into the ODE.

$$e^{-x}(u'' - 2u' + u) + (x+2)e^{-x}(u' - u) + (1+x)e^{-x}u = 0$$

Multiply both sides by e^x and combine like-terms.

$$u'' + xu' = 0$$

This new equation is first order in u'. Make the substitution, $v = u'$, to solve it.

$$v' + xv = 0$$

This equation can be solved by separation of variables.

$$\frac{dv}{dx} = -xv$$

$$\frac{dv}{v} = -x\,dx$$

Integrate both sides.

$$\ln|v| = -\frac{1}{2}x^2 + C_1$$

Exponentiate both sides.

$$|v| = e^{-\frac{1}{2}x^2 + C_1}$$

$$v(x) = \pm e^{C_1}e^{-\frac{1}{2}x^2}$$

Introduce a new constant.

$$v(x) = D_1 e^{-\frac{1}{2}x^2}$$

Now that we know $v(x)$, we can solve for $u(x)$.

$$u' = D_1 e^{-\frac{1}{2}x^2}$$

Integrate both sides with respect to x.

$$u(x) = \int_0^x D_1 e^{-\frac{1}{2}s^2}\, ds + D_2$$

The lower limit of integration is chosen to be 0 here, but know that it is arbitrary. D_2 would be adjusted to account for whatever lower limit was chosen when applying the initial conditions. Now that we know $u(x)$, we can solve for $y(x)$, the solution we care about.

$$y(x) = e^{-x} u(x)$$

Thus,

$$y(x) = e^{-x}\left(D_1 \int_0^x e^{-\frac{1}{2}s^2}\, ds + D_2\right),$$

where D_1 and D_2 are arbitrary constants. Note that the first solution we found by inspection, e^{-x}, is included in the answer for $y(x)$. This integral solution for $y(x)$ is perfectly acceptable, but it can also be expressed in terms of the error function $\operatorname{erf} x$, a known special function, which is defined as

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds.$$

Make the substitution,

$$\begin{aligned} r &= \frac{s}{\sqrt{2}} & \rightarrow \quad r^2 &= \frac{s^2}{2} \\ dr &= \frac{ds}{\sqrt{2}} & \rightarrow \quad \sqrt{2}\, dr &= ds, \end{aligned}$$

inside the integral in order to make the exponent the same as in the definition.

$$y(x) = e^{-x}\left(D_1 \sqrt{2} \int_0^{x/\sqrt{2}} e^{-r^2}\, dr + D_2\right)$$

Make it so the proper factor is in front of the integral.

$$y(x) = e^{-x}\left(D_1 \frac{\sqrt{2\pi}}{2} \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-r^2}\, dr + D_2\right)$$

Now we can write the solution in terms of the error function.

$$y(x) = e^{-x}\left[D_1 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + D_2\right]$$

Since D_1 and D_2 are arbitrary constants, $\sqrt{\pi/2}$ can be discarded by introducing new arbitrary constants, A and B. Therefore, another way to write the solution is

$$y(x) = e^{-x}\left[A \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + B\right].$$